OPTI 310, S. 2020
HW 4 solutions
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1 a) • $\bar{D} = D_x\hat{i} + D_y\hat{j} + D_z\hat{k}$ = Electric Displacement Field

$$\bar{D} = \varepsilon_0\bar{E} + \bar{P} \equiv \varepsilon_0\varepsilon_r\bar{E} = \varepsilon_0(1+\chi_e)\bar{E}$$

$\bar{P} = \varepsilon_0\chi_e\bar{E}$,

$n^2 = \varepsilon_r = 1 + \chi_e$

*where:
ρ_f = free charge Density
χ_e = electric susceptibility
ε_r = relative permitivity

ME#1 (i) $\boxed{\bar{\nabla}\cdot\bar{D} = \rho_f \Rightarrow \partial_x D_x + \partial_y D_y + \partial_z D_z = \rho_f}$

• $\bar{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ = Magnetic Induction

$\bar{B} = \mu_0\bar{H} + \bar{M}$ *for non magnetic $\bar{M}=0$; $\chi_m = 0$, $\mu_r = 1$

*assume $\bar{M}=0$ ⇒ $\bar{B} = \mu_0\bar{H}$

ME#4 (ii) $\boxed{\bar{\nabla}\cdot\bar{B} = 0 \Rightarrow \partial_x B_x + \partial_y B_y + \partial_z B_z = 0}$

ME#3 (iii) $\bar{\nabla}\times\bar{E} = -\partial_t\bar{B}$ *for non magnetic $\bar{B} = \mu_0\bar{H}$; $\bar{M}=$

$$\boxed{-\mu_0\partial_t(H_x\hat{i}+H_y\hat{j}+H_z\hat{k}) = -\partial_t(\mu_0\bar{H}) = (\partial_y E_z - \partial_z E_y)\hat{i} + (\partial_z E_x - \partial_x E_z)\hat{j} + (\partial_x E_y - \partial_y E_x)\hat{k}}$$

ME#2 (iv) $\bar{\nabla}\times\bar{H} = \bar{J}_{free} + \partial_t\bar{D}$

$$\boxed{(\partial_y H_z - \partial_z H_y)\hat{i} + (\partial_z H_x - \partial_x H_z)\hat{j} + (\partial_x H_y - \partial_y H_x)\hat{k} = (J_x\hat{i} + J_y\hat{j} + J_z\hat{k}) + \partial_t(D_x\hat{i} + D_y\hat{j} + D_z\hat{k})}$$

For all problems in HW set 4
3D space, & time
$f(\bar{r},t)$

* Note $\left\{ \begin{array}{l} \bar{D} = \bar{D}(\bar{r},t),\ \bar{E} = E(\bar{r},t),\ \bar{P} = \bar{P}(\bar{r},t) \\ \bar{B} = \bar{B}(\bar{r},t),\ \bar{H} = \bar{H}(\bar{r},t),\ \bar{M} = \bar{M}(\bar{r},t) \end{array} \right\}$ vector fields

• all fields are functions of space & time vectors
• Likewise, all sources are functions of space & time

$\left\{ \begin{array}{l} \rho = \rho(\bar{r},t) \text{ (scalar)} \\ \bar{J}_f = \bar{J}(\bar{r},t) \text{ (vector)} \end{array} \right\}$ sources

Remember Faraday & Ampere Equations have vectors on LHS & RHS include direction/unit vectors

2pts (1) b) Derive Wave Eqsn for $\bar{B}(\bar{r},t)$
assume $\rho_f = 0$, $\bar{J}_f = 0$ (no free charge or current)

Define • Maxwell's Equations (in "source free" Material)
no free charge - static or dynamic)

(1) $\bar{\nabla}\cdot\bar{D} = \cancel{\rho_f} \Rightarrow \bar{\nabla}\cdot\bar{D} = 0$ $\quad \bar{\nabla}\cdot\varepsilon_0\varepsilon_r\bar{E} = 0$

(2) $\bar{\nabla}\times\bar{E} = -\partial_t\bar{B}$ $\quad \bar{\nabla}\times\bar{E} = -\partial_t(\mu_0\mu_r\bar{H})$

(3) $\bar{\nabla}\times\bar{H} = \cancel{\bar{J}_f} + \partial_t\bar{D} \Rightarrow \bar{\nabla}\times\bar{H} = \partial_t\bar{D}$: $\quad \bar{\nabla}\times\bar{H} = \partial_t\varepsilon_0\varepsilon_r\bar{E}$

(4) $\bar{\nabla}\cdot\bar{B} = 0$ $\quad \bar{\nabla}\cdot\mu_0\mu_r\bar{H} = 0$

(★) $\bar{D} = \varepsilon_0\varepsilon_r\bar{E}$; $\bar{B} = \mu_0\bar{H} + M = \mu_0\mu_r\bar{H} \Rightarrow \bar{H} = (\mu_0\mu_r)^{-1}\bar{B}$

note* in non magnetic materials, $\mu_r = 1$ *

Solve • Want $\left(\bar{\nabla}^2 + \frac{1}{c^2}\partial_{tt}\right)\bar{B} = 0$ Form, but in material

Step (i) Using (★) Eliminate $\bar{H}$ & substitute $\bar{B}$ $\quad \bar{H} = [\mu_0\mu_r]^{-1}\bar{B}$

& Eliminate $\bar{D}$ & substitute $\bar{E}$

in all of Maxwells Eqsns:

(1) $\bar{\nabla}\cdot\varepsilon_0\varepsilon_r\bar{E}$ (2) $\bar{\nabla}\times\bar{E} = -\partial_t\bar{B}$ (3) $\bar{\nabla}\times[\mu_0\mu_r]^{-1}\bar{B} = \partial_t(\varepsilon_0\varepsilon_r\bar{E})$

(4) $\bar{\nabla}\cdot\bar{B} = 0$

Step (ii) Take the curl of the curl equation of the same field we want ($\bar{B}$ here) for our wave equation (Ampere's Law here)

$\bar{\nabla}\times(3)$

$$\bar{\nabla}\times\left(\bar{\nabla}\times(\mu_0\mu_r)^{-1}\bar{B}\right) = \bar{\nabla}\times\left(\partial_t\varepsilon_0\varepsilon_r\bar{E}\right) = \varepsilon_0\varepsilon_r\partial_t\{\bar{\nabla}\times\bar{E}\}$$

(iii) Sub. in curl equation for opposite field (2)
(here (2) Faraday's Law, which has Electric Field)
to Eliminate E from the equation
(since we want a resulting equation of only $\bar{B}$ mag. field

$$\underbrace{\boxed{\bar\nabla\times\bar\nabla\times[(\mu_0\mu_r)^{-1}\bar B]}}_{LHS}=\underbrace{\boxed{\varepsilon_0\varepsilon_r\partial_t\{\bar\nabla\times\bar E\}}}_{RHS}$$

RHS: $= \varepsilon_0\varepsilon_r\partial_t\{\bar\nabla\times\bar E\}=\boxed{\varepsilon_0\varepsilon_r\partial_t\{-\partial_t\bar B\}}$

Step (iv) Sub vector identity on left Hand side of equation (LHS)

$$\bar\nabla\times(\bar\nabla\times\bar A)=\bar\nabla(\bar\nabla\cdot\bar A)-\bar\nabla^2\bar A$$

so $\nabla\times\nabla\times\bar B=\bar\nabla(\bar\nabla\cdot\bar B)-\bar\nabla^2\bar B$

Step (V) Apply (4) Gauss's Law for Magnetic Fields $\bar\nabla\cdot\bar B=0$, to LHS result from step (iv) & RHS result from step (iii)

$$\underbrace{-\bar\nabla^2(\mu_0^{-1}\mu_r^{-1}\bar B)}_{(LHS)}=\underbrace{-\partial_t^2(\varepsilon_0\varepsilon_r\bar B)}_{(RHS)}$$

Step (vi) Set = to 0, & Factor out field

$$(\bar\nabla^2\mu_0^{-1}\mu_r^{-1}-\partial_t^2\varepsilon_0\varepsilon_r)\bar B=0$$

Step (vii) using $c=\frac{1}{\sqrt{\mu_0\varepsilon_0}}$ & assuming μ_0,μ_r are constants

$c^2=\mu_0\varepsilon_0$, Mult. both sides by $\mu_r\mu_0$ & rearrange

$$(\bar\nabla^2-(\mu_0\varepsilon_0)(\mu_r\varepsilon_r)\partial_t^2)\bar B=0$$

$$(\bar\nabla^2-\frac{n^2}{c^2}\partial_t^2)\bar B=\boxed{(\bar\nabla^2-\frac{1}{v^2}\partial_t^2)\bar B(\bar r,t)=0}$$

Notice
- $(\mu_r\varepsilon_r)=n^2$ is for being in material
- $(\mu_0\varepsilon_0)=\frac{1}{c^2}$ is for vaccum or free space

((sidenote: Set $\mu_r\approx 1$ usually in optics, & $n=\sqrt{\varepsilon_r}$; $\varepsilon_r=n^2$))

so $(\mu_0\varepsilon_0)\varepsilon_r\mu_r=\frac{n^2}{c^2}$ $n=\frac{c}{v}$ & $\frac{1}{v}=\frac{n}{c}$

1.c) $\bar{D} = \varepsilon_0 \varepsilon_r \bar{E}$, $\bar{B} = \mu_0 \mu_r \bar{H}$

- Harmonic Solution $\bar{B}(\bar{r},t) = \hat{k} B_0 e^{i(\bar{k}\cdot\bar{r} - \omega t)} = \underbrace{\hat{k} B_0 e^{i\bar{k}\cdot\bar{r}}} \underbrace{e^{i\omega t}}$

- assume from (b) $\rho_f = 0$, $\bar{J}_f = 0$ $\qquad \bar{B}(\bar{r},t) = \bar{B}(\bar{r}) \cdot \bar{B}(t)$

- Thus: Maxwell's Equations become:

(1) $\nabla \cdot \varepsilon \varepsilon_r \bar{E} = 0$ $\qquad \nabla \cdot \mu_0 \mu_r \bar{B} = 0$ (3)

(2) $\nabla \times \bar{E} = -\partial_t \bar{B}$ $\qquad \nabla \times \bar{H} = \partial_t \bar{D}$ (4)

- **(i)** show $\left(\nabla^2 - \frac{1}{v^2}\partial_t^2\right)\bar{B}(\bar{r},t) = 0$ $\quad$ where $\frac{1}{v^2} = \frac{n^2}{c^2} = \mu_0 \mu_r \varepsilon_0 \varepsilon_r$

$\mu_r = 1$ here

ie) $\nabla^2 \bar{B}(\bar{r},t) = \frac{1}{v}\partial_t^2 \bar{B}(\bar{r},t)$ $\quad$ since $\mu_r \varepsilon_r = n^2$

$\mu_0 \varepsilon_0 = \frac{1}{c^2}$

$\nabla^2 \bar{B}(\bar{r},t) =$ $\qquad (i^2 = -1)$

- $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\bar{B}(\bar{r},t) = \left(\partial_x^2 \hat{k} B_0 e^{i\bar{k}\cdot\bar{r}} + \partial_y^2 \hat{k} B_0 e^{i\bar{k}\cdot\bar{r}} + \partial_z^2 \hat{k} B_0 e^{i\bar{k}\cdot\bar{r}}\right) e^{i\omega t}$

$= \left(-k_x^2 + -k_y^2 + -k_z^2\right)\bar{B}(\bar{r},t) = \frac{n^2}{c^2}\partial_t^2 B(\bar{r},t) = \frac{n^2}{c^2}\left(+ \cdot - \omega^2\right)\bar{B}(\bar{r},t)$

$-\bar{k}^2 \bar{B}(\bar{r},t) = \frac{n^2}{c^2}(-\omega^2)\bar{B}(\bar{r},t)$

$+\bar{k}^2 = +\frac{n^2}{c^2}\omega^2$ $\qquad \boxed{|\bar{k}| = \frac{n\omega}{c}}$ ✓

- **(ii)** $\bar{\nabla} \cdot \bar{B}(\bar{r},t) = 0 \Longrightarrow \nabla \cdot \hat{k} B_0 e^{i(\bar{k}\cdot\bar{r} - \omega t)} = 0$

$\hat{\imath} \cdot \hat{k} = \hat{\jmath} \cdot \hat{k} = 0$, $\hat{k} \cdot \hat{k} = 1$ $\quad \hat{k}$ is unit vector.

$\partial_z^2 \hat{k} \cdot \hat{k}\bar{B}(\bar{r},t) = \partial_z^2 B_0^{\,i(\bar{k}\cdot\bar{r} - \omega t)} = 0$ $\quad$ only non-zero Dot Product on 2 HS Still must $= 0$!

$$\boxed{\nabla \cdot \bar{B}(\bar{r},t) = i\bar{k} \cdot \hat{k} B_0 e^{i(\bar{k}\cdot\bar{r} - \omega t)} = 0}$$

implies $\hat{k} \perp \bar{k}$, the Magnetic field oscillates in a vector direction $\hat{k}$ that is transverse to $\bar{k}$ propagation directi

1d) $J_f = \rho_f$ same maxwells eqsn from part (c), $\bar{H} = (\mu_0\mu_r)^{-1}\bar{B}$

$$\bar{\nabla} \times [\bar{H}] = -\partial_t[\bar{D}] \implies \bar{\nabla} \times [(\mu_0\mu_r)^{-1}\bar{B}] = -\partial_t[\varepsilon_0\varepsilon_r\bar{E}]$$

$$\Rightarrow \bar{\nabla} \times \bar{B} = -(\mu_0\mu_r\varepsilon_0\varepsilon_r)\partial_t\bar{E} = \frac{-1}{V^2}\partial_t\bar{E}$$

For $\mu_r\varepsilon_r = n^2$, $\mu_0\varepsilon_0 = \frac{1}{c^2}$ & $n = \frac{c}{v}$

LHS

$$\bar{\nabla} \times \bar{B} = [\hat{i}(\partial_y B_z - \cancel{\partial_z B_y}) + \hat{j}(\cancel{\partial_z B_x} - \partial_x B_z) + \hat{k}(\cancel{\partial_x B_y} - \cancel{\partial_y B_x})]e^{-i\omega t}$$

Where $B_x = B_y = 0$ for Magnetic field given in part (c)

$$\Rightarrow \bar{\nabla} \times \bar{B} = [\hat{i}\,\partial_y B_z + \hat{j}(-\partial_x B_z)]e^{-i\omega t}$$

$$= [\hat{i}(ik_y)B_z + \hat{j}(-ik_z)B_z]e^{-i\omega t}$$

putting together RHS = LHS & $\bar{B}(\bar{r},t) = \bar{B}(\bar{r})e^{-i\omega t}$; $\bar{E}(\bar{r},t) = \bar{E}(\bar{r})e^{-i\omega t}$ Harmonic Fields

$$\frac{-1}{V^2}\partial_t(\bar{E}(\bar{r})e^{-i\omega t}) = \frac{+1}{V^2}(+i\omega)\bar{E}(\bar{r})e^{-i\omega t}$$

$$\frac{i\omega}{V^2}\bar{E}(\bar{r})\cancel{e^{-i\omega t}} = i(k_y B_z\hat{i} - k_z B_z\hat{j})\cancel{e^{-i\omega t}}$$

$$\boxed{\bar{E}(\bar{r},t) = \frac{V^2}{\omega}(k_y B_z\hat{i} + k_x B_z\hat{j})e^{-i\omega t}} = E(r)e^{-i\omega t}$$

1e) From $\bar{\nabla}\cdot\bar{E} = 0$ we know $k_x E_x + k_y E_y + k_z E_z = 0$

Wave prop. down x axis

Since $\bar{K} \perp \bar{B}$ & $\bar{K} \perp \bar{E}$ & the cross product in Ampere's Law tells us that $\bar{B} \perp \bar{E}$

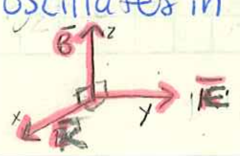
We form a right hand coordinate system $\bar{K} \perp \bar{B} \perp \bar{E}$

now $\Rightarrow$ Wave propagates along x axis $\Rightarrow \bar{K} = K\hat{i}$; $K_x E_x = K_x B_x = 0$

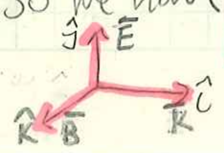
Since $E(\bar{r}) = \frac{+V^2}{\omega}K_x B_z\hat{j}$ because $K_y = K_z = 0$ and $B \perp E$ from Ampere's Law

* $\bar{E}$ oscillates in y-direction *, so we have $\boxed{\bar{E}(\bar{r},t) = \hat{j}E_y e^{-i\omega t};\ \bar{B}(\bar{r},t) = \hat{k}B_z e^{-i\omega t}\ \&\ \bar{K} = K_x\hat{i}}$

see (pg 13) in notes



or we can draw it



② $\bar{S} = \bar{E} \times \bar{H}$ $\quad \bar{B} = \mu_0 \mu_r \bar{H}$ for $\mu_r = 1$ $\quad \bar{H} = \frac{1}{\mu_0}\bar{B}$

$\Rightarrow \bar{S} = \frac{1}{\mu_0}(\bar{E} \times \bar{B})$: The Poynting Vector

ⓐ For Harmonic wave propagating along the z-axis: $\bar{K} = K_z \hat{K}$

$$\bar{S} = \frac{1}{\mu_0}\sin(Kz - \omega t + \varepsilon)\underbrace{[\hat{\imath} E_0 \times \hat{\jmath} B_0]}$$

cross product yeilds a vector ⊥ to the vectors being crossed.

Since $\bar{E}$ is osillating in x-direction (only $\hat{\imath}$ compound given)

& $\bar{B}$ is osillating in y-direction (only $\hat{\jmath}$ compound given)

$\boxed{\text{Here: } \hat{\imath} \times \hat{\jmath} = \hat{K}}$

We expect propagation direction in $+\hat{z}$ axis, so $\bar{K} = |\bar{K}|\hat{K}$, & $\bar{S} = S\hat{K}$

$$\bar{S} = \frac{1}{\mu_0}\sin(Kz - \omega t + \varepsilon)[(E_0 B_0)\hat{K}]$$

Since $E_0 = v B_0 \Rightarrow B_0 = \frac{E_0}{v}$ *Eliminate B_0*

$$\bar{S} = \frac{1}{\mu_0}\sin(kz - \omega t + \varepsilon)\left(\frac{E_0^2}{v}\right)\hat{K}$$

& $E_0^2 = |\bar{E}_0|^2$ here

$$\langle\bar{S}\rangle_t = \frac{E_0^2}{\mu_0 v}\hat{K}\left[\frac{\int_t^{t+\Delta T}\sin(k_z - \omega t + \varepsilon)\,dT}{\Delta T}\right] = \frac{E_0^2}{\mu_0 v}\hat{K}\underbrace{\left[\frac{1}{\Delta T}\left(\frac{\Delta T}{2} - (0)\right)\right]}_{=\frac{1}{2}}$$

$$\langle\bar{S}\rangle_t = \frac{1}{2}\frac{E_0^2}{\mu_0 v}\hat{K} = \frac{1}{2}\frac{(E_0^2)}{(\mu_0(c/n))}\hat{K}$$

since $n = \frac{c}{v}$ $\quad v = \frac{c}{n}$

$\frac{1}{v} = \frac{n}{c} = \sqrt{\mu_0 \varepsilon_0 \mu_r \varepsilon_r}$

as expected time average of a sine function is $\frac{1}{2}$

$$\boxed{\langle\bar{S}\rangle_t = \frac{1}{2}\{\varepsilon_0 n c E_0^2\}\hat{K}}$$

$$\boxed{I = \langle S \rangle_T}$$

The Magnitude of the energy flow $I = \frac{n}{2}\varepsilon_0 c|E_0|^2$

& is propagating in the $\hat{K}$ direction.

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2b) $I = \frac{\text{Total Power}}{\text{Total Area}} = \frac{1[KW]}{1[cm][cm]} = \frac{1\times 10^3 [W]}{10^{-2}[m]10^{-2}[m]} = 10^3 \times 10^4 \left[\frac{W}{m^2}\right]$

$I = 10^7 \left[\frac{W}{m^2}\right] = \frac{1}{2} c \varepsilon_0 E_0^2$

in Air / Free space: $n = 1$, $\varepsilon_0 = 8.854 \times 10^{-12} \left[\frac{\text{farad}}{\text{meter}}\right]$

$E_0 = \sqrt{\frac{2}{c\varepsilon_0} I} = \sqrt{\frac{2(10^7 [W/m^2])}{(3\times 10^8 [m/s]) 8.854 \times 10^{-12} \left[\frac{\text{farad}}{m}\right]}} \Rightarrow \boxed{E_0 = 8.6802 \times 10^4 \left[\frac{V}{m}\right]}$

$\boxed{B_0 = \frac{E_0}{c} = 2.8954 \times 10^{-4} [T]}$

• $h = 6.626 \times 10^{-34} [Js]$
$= 4.135 [eV \cdot s]$

c) $\varepsilon = \hbar\omega = \left(\frac{h}{2\pi}\right)\left(2\pi\nu\right) = h\nu = \frac{hc}{\lambda}$

$\left[\begin{array}{c} \lambda\nu = c \\ \omega = 2\pi\nu \\ \hbar = \frac{h}{2\pi} \end{array}\right]$

$\varepsilon [eV] \approx \frac{1.24 [eV\text{-}\mu m]}{\lambda [\mu m]} = \frac{1.24}{10} [eV]$

• $\hbar = 1.054 \times 10^{-34} [J\cdot s]$
$= 6.582 \times 10^{-16} [eVs]$

$\boxed{\varepsilon \approx 0.124 [eV] \quad \text{or} \quad \varepsilon \approx 1.987 \times 10^{-20} [J]}$ this is the energy $\left[\frac{\text{Joules}}{\text{photon}}\right]$ per photon

• $\text{Total Power} = \frac{\text{Total Energy}}{\text{Second}} = \frac{\Sigma \varepsilon_p}{\text{sec.}}$ where $\Sigma\varepsilon_p$ = sum of energy from each photon.

$R = 10^3 \left[\frac{\text{Joules}}{\text{Second}}\right] \left(\frac{1}{1.987\times 10^{-20}} \left[\frac{1}{\frac{\text{Joules}}{\text{photon}}}\right] \right)$

$= \frac{10^3}{1.987 \times 10^{-20}} \left[\frac{\text{Photons}}{\text{Second}}\right] =$

$\boxed{R = 5.033 \times 10^{22} \left[\frac{\text{photons}}{\text{second}}\right]}$

d) $p = \frac{h}{\lambda} = \frac{hc/\lambda}{c} = \frac{\varepsilon_p}{c} = \frac{1.987 \times 10^{-20}[J]}{3.0 \times 10^8 [m/s]}$ $\quad 6.623 \times 10^{-29} \left[\frac{J}{m/s}\right]$

$\boxed{p = 6.623 \times 10^{-29} \left[\frac{kg\, m}{s}\right]}$ ✓

$\frac{Js}{m} = \frac{kg\,m}{s}$ ✓ correct units of momentum!! :)

Where p is the magnitude of the momentum of a single photon of energy $\varepsilon_p \approx 0.124 [eV]$